5.6.1 Kinematics Toolkit

Displacement, Velocity & Acceleration

What is kinematics?

- **Kinematics** is the branch of mathematics that models and analyses the **motion** of objects.
- Common words such as **distance**, **speed** and **acceleration** are used in kinematics but are used according to their technical definition.

What definitions do I need to be aware of?

Firstly, only motion of an object in a **straight line** is considered:

- this could be a **horizontal** straight line — the **positive** direction is to the **right**;
- or this could be a **vertical** straight line — the **positive** direction is **upwards**.

Particle

- A **particle** is the general term for an object; some questions may use a specific object such as a car or a ball.

Time t seconds

- Displacement, velocity and acceleration are all **functions of time** t.
- **Initially** time is zero: $t = 0$.

Displacement s m

- The **displacement** of a particle is its distance relative to a **fixed point**; the fixed point is often (but not always) the particle's **initial position**.
- Displacement will be **zero**, $s = 0$, if the object is at or has returned to its initial position.
- Displacement will be **negative** if its position relative to the fixed point is in the negative direction (left or down).

Distance d m

- Use of the word **distance** needs to be considered carefully and could refer to:
  - the distance **travelled** by a particle;
  - the (straight-line) distance the particle is from a particular point.
- Be careful not to confuse **displacement** with **distance**: if a bus route starts and ends at a bus depot, when the bus has returned to the depot its displacement will be zero but the distance the bus has travelled will be the length of the route.
- **Distance is always positive.**

Velocity $v \text{ m s}^{-1}$

- The **velocity** of a particle is the rate of change of its displacement at time t.

- Velocity will be **negative** if the particle is moving in the negative direction.
- A velocity of **zero** means the particle is **stationary**: $v = 0$.

Speed $|v|\ \mathrm{m\,s^{-1}}$

- **Speed** is the magnitude (absolute value or modulus) of velocity; as the particle moves in a straight line, speed is the velocity ignoring direction:
  - if $v = 4$, then $|v| = 4$;
  - if $v = -6$, then $|v| = 6$.

Acceleration $a\ \mathrm{m\,s^{-2}}$

- The **acceleration** of a particle is the rate of change of its velocity at time t.
- Acceleration can be negative, but this alone cannot fully describe the particle's motion:
  - if velocity and acceleration have the **same sign**, the particle is **accelerating** (speeding up);
  - if velocity and acceleration have **different signs**, the particle is **decelerating** (slowing down);
  - if acceleration is zero, $a = 0$, the particle is moving with **constant velocity**;
  - in all cases the **direction of motion** is determined by the sign of the velocity.

Are there any other words or phrases in kinematics I should know?

Certain words and phrases imply values or directions:

- A particle described as "at rest" means its velocity is zero: $v = 0$.
- A particle moving "due east" or "right" is moving in the positive horizontal direction, so $v > 0$.
- A particle "dropped from the top of a cliff" or moving "down" is in the negative vertical direction, so $v < 0$.

What are the key features of a velocity–time graph?

- The **gradient** of the graph equals the **acceleration** of the object.
- A **straight line** shows constant acceleration.
- A **horizontal line** shows constant velocity.
- The **area** between the graph and the t-axis gives the **change in displacement**:
  - graph **above** the axis $\Rightarrow$ moving **forwards**;
  - graph **below** the axis $\Rightarrow$ moving **backwards**.
- **Total displacement** from the starting point = (sum of areas above axis) − (sum of areas below axis).
- **Total distance travelled** = sum of *all* areas (above and below).

- If the graph **touches** the t-axis, the object is **stationary** at that instant.

Examiner Tip

In an exam, if you are given an expression for the velocity, sketching a velocity–time graph can help you visualise the problem.

Worked Example

A particle is projected vertically upwards from ground level, taking 8 seconds to return to the ground. The velocity–time graph shows the motion of the particle.

(i) How many seconds does the particle take to reach its maximum height? Give a reason.
Solution: At maximum height the velocity is zero. From the graph, $v = 0$ at $t = 4$.

$\therefore$ The particle takes **4 seconds** to reach its maximum height, because $v = 0\ \text{m s}^{-1}$ at $t = 4$.

(ii) State, with a reason, whether the particle is accelerating or decelerating at $t = 3$.
Solution: At $t = 3$, the velocity is **positive** (graph is above the axis). The gradient (acceleration) is **negative** (the line slopes downward). Since velocity and acceleration have different signs, the particle is **decelerating**.

$\therefore$ At $t = 3$ the particle is **decelerating**, because its velocity and acceleration have different signs.

5.6.2 Calculus for Kinematics

Differentiation for Kinematics

How is differentiation used in kinematics?

Displacement, velocity and acceleration are related by calculus. In terms of differentiation:

- **Velocity** is the rate of change of displacement:

$$v = \frac{ds}{dt} \qquad \text{or} \qquad v(t) = s'(t).$$

- **Acceleration** is the rate of change of velocity:

$$a = \frac{dv}{dt} \qquad \text{or} \qquad a(t) = v'(t).$$

- So acceleration is also the **second derivative** of displacement:

$$a = \frac{d^2s}{dt^2} \qquad \text{or} \qquad a(t) = s''(t).$$

If a graph is not given you can use a GDC to draw one and find gradients:

- velocity is the gradient on a displacement–time graph;
- acceleration is the gradient on a velocity–time graph.

Worked Example

The displacement, s m, of a particle at t seconds is modelled by

$$s(t) = 2t^3 - 27t^2 + 84t.$$

(i) Find $v(t)$ and $a(t)$.

$$v(t) = s'(t) = 6t^2 - 54t + 84 = 6(t^2 - 9t + 14)$$
$$\therefore v(t) = 6(t-7)(t-2)$$
$$a(t) = v'(t) = 12t - 54 = 6(2t-9)$$

(ii) Find the times at which the particle is at rest.
The particle is at rest when $v(t) = 0$:

$$6(t-7)(t-2) = 0 \implies t = 2 \text{ or } t = 7.$$

$\therefore$ The particle is at rest at $t = 2$ seconds and $t = 7$ seconds.

Integration for Kinematics

How is integration used in kinematics?

Since $v = \dfrac{ds}{dt}$, it follows that

$$s = \int v\,dt.$$

Similarly, velocity is an antiderivative of acceleration:

$$v = \int a\,dt.$$

How do I find the constant of integration?

A **boundary** or **initial condition** must be known:

- The word "initially" or "initial" refers to $t = 0$.
- Other information may be given at a specific time (a **boundary condition**).
- Substituting the known values allows the constant c to be found.

How are definite integrals used in kinematics?

- The **displacement** of a particle between times $t = t_1$ and $t = t_2$:

  $$\int_{t_1}^{t_2} v(t)\,dt.$$

  This equals (total area above t-axis) $-$ (total area below t-axis).

- The **distance travelled** between times $t = t_1$ and $t = t_2$:

$$\int_{t_1}^{t_2} |v(t)|\, dt.$$

This equals the sum of *all* areas (above and below the axis). Use a GDC to plot $y = |v(t)|$.

Examiner Tip

Sketching the velocity–time graph can help you visualise distances travelled using areas between the graph and the horizontal axis.

Worked Example

A particle moving in a straight horizontal line has velocity $v(t) = 8t^3 - 12t^2 - 2t \text{ m s}^{-1}$ at time t seconds.

(i) Given that the initial position of the particle is at the origin, find $s(t)$.

$$s(t) = \int v(t)\, dt = \int \left(8t^3 - 12t^2 - 2t\right) dt = 2t^4 - 4t^3 - t^2 + c.$$

At $t = 0$, $s = 0$, so $c = 0$.

$$\therefore s(t) = 2t^4 - 4t^3 - t^2.$$

(ii) Find the displacement from the origin in the first five seconds.

$$s = \int_0^5 \left(8t^3 - 12t^2 - 2t\right) dt = \left[2t^4 - 4t^3 - t^2\right]_0^5 = 725 \text{ m}.$$

(iii) Find the distance travelled in the first five seconds.

$$d = \int_0^5 |v(t)|\, dt = \int_0^5 \left|8t^3 - 12t^2 - 2t\right| dt \approx 736.73\ldots \approx 737 \text{ m (3 s.f.)}.$$